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2000 J. Phys. A: Math. Gen. 33 3989

(http://iopscience.iop.org/0305-4470/33/22/303)

J. Phys. A: Math. Gen. **33** (2000) 3989–3995. Printed in the UK PII: S0305-4470(00)10929-1

Exactly solvable statistical model for two-way traffic

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Received 7 January 2000

Abstract. We generalize a recently introduced traffic model, where the statistical weights are associated with whole trajectories, to the case of two-way flow. An interaction between the two lanes is included which describes a slowing down when two cars meet. This leads to two coupled five-vertex models. It is shown that this problem can be solved by reducing it to two one-lane problems with modified parameters. In contrast to stochastic models, jamming appears only for very strong interaction between the lanes.

1. Introduction

The non-equilibrium properties of one-dimensional lattice gases have been studied intensively over the last years [1]. With lattice gases, one can model not only physical situations such as transport in solid ionic conductors [2] or growth processes [3], but also the traffic flow on roads [4]. Moreover, they can be used to study general features of phase transitions in non-equilibrium systems [5–8]. For the traffic problem, the simplest model is the completely asymmetric exclusion process (ASEP), where classical hard-core particles hop stochastically, with unit rate, in one direction only [9]. On a ring, one then finds a steady state of product form where all configurations are equally likely. In terms of the density ρ of particles, the flux is then given by $j = \rho(1 - \rho)$ and already shows the qualitative features also found in more sophisticated models, i.e. it vanishes for $\rho = 0$, 1 and has a maximum in between.

An essentially new description of traffic flow was proposed recently by Brankov *et al* [10]. In this work, non-intersecting domain-wall lines on a square lattice were interpreted as space–time trajectories of cars. The weight of a trajectory is then obtained from the fugacities for horizontal and vertical moves. The single step, however, has no stochastic interpretation. The problem can be formulated in terms of a five-vertex model which generates these lines and which is exactly solvable since it satisfies the so-called free-fermion condition. The result for the flux j is physically reasonable and very similar to that for a variant of the (stochastic) Nagel–Schreckenberg model [11]. In this paper, we show that one can generalize this model to the case of two-way traffic where cars on different lanes interact with each other. The specific effect which we are treating is a tendency to slow down when another car is approaching. In the two-dimensional formulation, this is described by a modification of the fugacities whenever trajectories of oppositely moving cars cross. One then is led to consider two five-vertex models with a certain coupling between them. It turns out, however, that this coupling only renormalizes the parameters in each subsystem, so the problem remains solvable as before. One finds that, in this model, the effect of an obstacle, i.e. of a car in the other lane, is relatively

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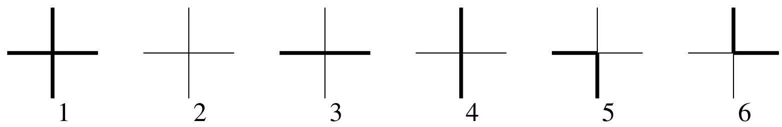

Figure 1. Vertex configurations for right-moving cars: Boltzmann weights are 0, 1, x_1, t_1, $\sqrt{x_1 t_1}$ and $\sqrt{x_1 t_1}$ respectively.

weak. While in stochastic models already a certain finite reduction of the hopping rate at one position usually leads to a traffic-jam phenomenon with a region of high density appearing in front of the bottleneck [12–17], this happens here only if the fugacity is reduced to zero for a large system. As will be explained, this feature is related to the different weighting of the trajectories in both cases.

In the following, we first describe the model in section 2 and then explain its solution in section 3. Finally, in section 4, we discuss the results and add some further remarks.

2. Model

We first recall the formulation of the original one-way traffic model in [10]. For a square lattice with periodic boundary conditions, the horizontal direction is interpreted as space, the vertical one as time (increasing downwards). Non-intersecting lines running towards the lower right are then drawn on the lattice and viewed as trajectories of right-moving cars. They do not end, so that the number N_1 of cars is conserved. A horizontal step, representing a move, is given fugacity (weight) x_1, and vertical step fugacity t_1. Statistical averages are then obtained from the partition function

$$Z(N_1, x_1, t_1) = \sum_C {x_1}^{N_x(C)} {t_1}^{N_t(C)} \qquad (1)$$

where $N_x(C)$ and $N_t(C)$ are the total numbers of steps in the two directions for a certain configuration C of trajectories. These trajectories are generated with their correct weights if at each lattice site the vertices shown in figure 1 are possible.

Since crossings (vertex 1) are forbidden, one is effectively dealing with a five-vertex model which can be solved exactly via the Bethe ansatz, even for more general weights w_5, w_6 [18, 19]. In the present case, free-fermion techniques can be used to obtain the partition function [20].

For the two-way traffic model, we introduce a second lattice where trajectories run towards the lower left, corresponding to the cars in the other lane. The fugacities are taken to be x_2, t_2 and the trajectories are now generated by the vertices in figure 2.

To formulate the interaction between cars in the two lanes, the indices at the vertices are specified in the following way:

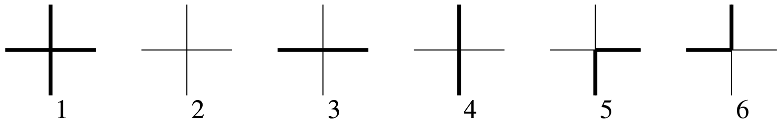

Figure 2. Vertex configurations for left-moving cars: Boltzmann weights are 0, 1, x_2, t_2, $\sqrt{x_2 t_2}$ and $\sqrt{x_2 t_2}$ respectively.

Table 1. Interaction ϵ between two adjacent vertices in the two layers.

Vertex	3	4	5	6
3	0	$-h$	$-h/2$	$-h/2$
4	$-h$	0	$-h/2$	$-h/2$
5	$-h/2$	$-h/2$	$-h/2$	$-h/2$
6	$-h/2$	$-h/2$	$-h/2$	$-h/2$

The variables $\alpha, \beta, \ldots$ take the value one if a car is present (thick line) and zero otherwise. For all vertices, the so-called ice rule

$$\alpha + \beta = \alpha' + \beta' \qquad \text{and} \qquad \gamma + \delta = \gamma' + \delta' \tag{2}$$

holds, which ensures the conservation law for the number of cars, separately for both lanes.

We now imagine that the two lattices are placed above each other and attribute an additional Boltzmann weight

$$v = \exp(-\epsilon) = \exp\left(\frac{-h}{2}(\alpha\delta + \alpha'\delta' + \beta\gamma' + \beta'\gamma)\right) \tag{3}$$

to adjacent vertices in the two layers. Then each crossing of two trajectories will be weighted with the factor

$$0 < r = \exp(-h) < 1. \tag{4}$$

To see this, one first notes that $\epsilon = 0$, $v = 1$ if one of the vertices is of type 2, i.e. if there is no car present. The values of ϵ in the remaining cases are given in table 1. It then follows that simple crossings, which involve a pair of vertices of type 3 and 4, lead directly to a factor r (see table 1). If trajectories meet and run (anti)parallel before they separate again, each of the two branch points contributes a factor $\sqrt{r}$. Some examples illustrating such crossings are shown in figure 3.

One should mention that the choice (3) for the interaction is not unique. The more general form for ϵ

$$-\epsilon = A(\alpha\delta' + \beta\gamma) + B(\alpha\delta + \beta'\gamma) + C(\alpha'\delta' + \beta\gamma') + D(\alpha'\delta + \beta'\gamma') \tag{5}$$

with $A + B + C + D = h$ still leads to the same factor $r = \exp(-h)$. The individual terms listed in table 1, however, become more complicated.

In the model defined in this way, one still has the freedom to choose the particle numbers and the fugacities. Thus, by setting $x_2 = 0$, one can immobilize the cars in the second lane and treat, in particular, the case of one fixed obstacle, which is of special interest.

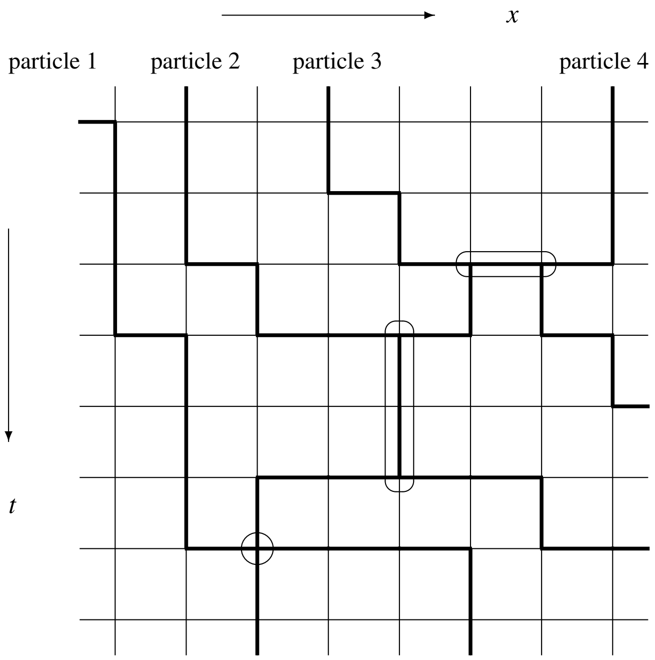

Figure 3. Trajectories of one left-moving and three right-moving particles with different types of crossing.

3. Solution

We now show that the two-lane model can be solved by reducing it to the one-lane problem. The proof follows [21], where a similar problem was treated. It is based on the ice rule (2) which relates horizontal and vertical bond variables. Suppose that the lattices have N columns and M rows, and let $\alpha_{n,m}$ ($\gamma_{n,m}$) and $\beta_{n,m}$ ($\delta_{n,m}$) be the variables to the right and below the vertex (n, m), respectively, in the two layers. Then the total interaction is

$$E = -\frac{h}{2}\sum_{m=1}^{M}\sum_{n=1}^{N}(\alpha_{n-1,m}\delta_{n,m-1} + \alpha_{n,m}\delta_{n,m} + \beta_{n,m-1}\gamma_{n-1,m} + \beta_{n,m}\gamma_{n,m}).$$

With the help of (2), this can be rewritten as

$$E = -\frac{h}{2}\left(\sum_{m=1}^{M}(\alpha_{0,m} + \alpha_{N,m})N_2 + \sum_{m=1}^{M}(\gamma_{0,m} + \gamma_{N,m})N_1 + U(0) - U(M)\right). \quad (6)$$

where

$$U(m) = \sum_{n=1}^{N-1}\sum_{k=1}^{n}(\beta_{k,m}\delta_{n+1,m} - \beta_{n+1,m}\delta_{k,m}) \quad (7)$$

contains only vertical bonds, while the other two terms in (6) contain only horizontal bonds. Due to the periodic boundary conditions, the difference $U(0) - U(M)$ vanishes and one obtains

$$E = -hN_2\sum_{m=1}^{M}\alpha_{N,m} - hN_1\sum_{m=1}^{M}\gamma_{N,m}. \quad (8)$$

This can be compared with the effect of a rescaling $x_1 \to x_1 e^{-\eta_1}$, $x_2 \to x_2 e^{-\eta_2}$ which leads to an extra factor $\exp(-\eta_1(\alpha+\alpha')/2 - \eta_2(\gamma+\gamma')/2)$ for each pair of adjacent vertices. Summed over all sites, this gives

$$E' = -\sum_{m=1}^{M}\sum_{n=1}^{N}\left(\frac{\eta_1}{2}(\alpha_{n-1,m}+\alpha_{n,m}) + \frac{\eta_2}{2}(\gamma_{n,m}+\gamma_{n-1,m})\right)$$

which can be expressed as

$$E' = -\sum_{m=1}^{M}\frac{\eta_1}{2}N(\alpha_{0,m}+\alpha_{N,m}) + V(0) - V(M) - \sum_{m=1}^{M}\frac{\eta_2}{2}N(\gamma_{0,m}+\gamma_{N,m}) - \tilde{V}(0) + \tilde{V}(M)$$

where

$$V(m) = \sum_{n=1}^{N-1}\sum_{k=1}^{n}\beta_{k,m} - \sum_{n=2}^{N}\sum_{k=n}^{N}\beta_{k,m} \tag{9}$$

and $\tilde{V}(m)$ is defined analogously with $\beta \to \delta$. Using again the periodic boundary conditions, one finds

$$E' = -\eta_1 N\sum_{m=1}^{M}\alpha_{N,m} - \eta_2 N\sum_{m=1}^{M}\gamma_{N,m} \tag{10}$$

which has the same form as E in (8). Therefore, the interaction has the same effect as a change in the horizontal fugacities if one chooses $\eta_1 = hN_2/N = h\rho_2$ and $\eta_2 = hN_1/N = h\rho_1$, where ρ_1 and ρ_2 are the densities of cars in the two lanes. The partition function is then

$$Z(N_1, N_2, x_1, x_2, t_1, t_2, r) = Z(N_1, x_1 r^{\rho_2}, t_1)Z(N_2, x_2 r^{\rho_1}, t_2). \tag{11}$$

This exact formula looks like the result of a mean-field treatment since only the densities in the other layer enter the expressions. One should point out that it also holds for more general choices of the vertex weights in the layers. Then, also the weight w_1 of vertex 1 has to be renormalized with the same exponential factor.

4. Results and discussion

One can now make use of the results for the single-lane case [10]. For one lane, the flux per site is equal to the average number of horizontal steps and given by

$$j(\rho, x) = \frac{\langle N_x\rangle}{NM} = \frac{1}{2}\left[\frac{1}{\pi}\arccos\left(\frac{c-2x+cx^2}{1-2xc+x^2} - 1\right) - \rho\right] \tag{12}$$

where $c = \cos(\pi\rho)$ and $x < 1$ has been assumed. This is the physical region since the average speed of one car is $v = x/(1-x)$. As a function of ρ, the flux has a maximum at $\rho = (1/\pi)\arccos(x)$, which shifts from $\rho = \frac{1}{2}$ to $\rho = 0$ as x increases.

By inserting $x_1 r^{\rho_2}$ and $x_2 r^{\rho_1}$ into (14), one then obtains the fluxes j_1 and j_2 in the two-lane case. These do not depend on the motion in the other lane, but only on the density there. Since j increases with x, the interaction factor r^{ρ_α} always reduces the flux, as expected. This reduction, however, becomes smaller as the density in the second lane decreases. For the case of only a single car one has

$$j_1 = j(\rho_1, x_1 r^{1/N}) \tag{13}$$

and this approaches the value $j(\rho_1, x_1)$ without interaction for large N. In order to slow down the traffic appreciably, one would need $h \approx N$, i.e. an interaction increasing with the size, so that r vanishes exponentially. In other words, a transition only occurs at $r = 0$.

As mentioned, the situation is different for stochastic models. There j shows a sudden decrease as soon as the corresponding quantity r (describing the reduced crossing probability at a defect) falls below a certain finite value r_c. This is connected with the appearance of a jam at the defect. In terms of trajectories, the effect can be described as follows. Consider a stochastic model as in [10] where a particle can move an arbitrary distance horizontally, at each step continuing with probability p and stopping with probability $q = 1 - p$. At the defect, the quantities are $p' < p$ and $q' > q$. A particle some steps away from the defect will typically move to the bottleneck and then stay there for some time. Due to $q' > q$, such a trajectory has a higher weight than any other one where it makes stops before and then crosses the defect immediately. The same holds for another particle following it, since this has $q' = 1$ once it has reached the site next to the first one. In this way, the jam builds as a region of vertical trajectories to the left of the defect.

In the present model, the picture is different. There is no advantage in staying at the blockage, the crossing factor r and the weights $x^k t^l$ are the same as for paths which approach the defect gradually. Nor is there an advantage for following particles to move next to the preceding one. Therefore no jam builds up. One could say that the model mimics the anticipation of disturbances by producing less densely packed trajectories. But, in shifting r_c to zero, it overestimates the effect.

It is also interesting to compare the two models at the operator level. According to [22], the transfer matrix T of the (one-layer) five-vertex model commutes with the operator

$$\mathcal{H} = -\sum_n (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + 2H\sigma_n^z) \tag{14}$$

where $H = (1 + x^2 - t^2)/2x$, and it is easy to see that the ground state of $\mathcal{H}$ gives the maximal eigenvalue of T. This operator shows very clearly the free-fermion character of the model and also its non-stochastic nature, since the necessary $\sigma^z\sigma^z$-terms (which are related to the loss processes in the master equation) are missing.

If one uses more general vertex weights w_5 and w_6, the operator

$$\mathcal{H} = -\sum_n (\sigma_n^- \sigma_{n+1}^+ + \Delta\sigma_n^z \sigma_{n+1}^z) \tag{15}$$

commutes with T, where $\Delta = (w_3 w_4 - w_5 w_6)/(w_2 w_4)$ [18, 23]. Although this contains such terms and has the form of the time-evolution operator for fully asymmetric hopping [24, 25], the fact that Δ is not equal to one still makes it different. On the other hand, this model is interesting, because it contains, in the x–t plane, a frozen phase with density $\rho = \frac{1}{2}$ [18, 19], where the trajectories have the form of stairs with steps of unit length in both directions. This corresponds to synchronized traffic with always one empty site between the cars. As this phase gives the highest possible throughput of vehicles and persists for a wide range of parameters x, t, it represents the analogue of the maximal current phase in stochastic hopping models [8, 26]. In the j–ρ relation, one then finds a cusp at $\rho = \frac{1}{2}$. As mentioned above, this model can also be treated in the two-way case. However, apart from half-filling, the blocking properties will be similar to those described above.

Acknowledgment

VP would like to thank the Alexander von Humboldt Foundation for financial support.

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